JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 26, No. 2, March–April 2003

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Structured Multivariable Phase Margin Analysis with Applications to a Missile Autopilot

Jonathan R. Bar-on*
Science Applications International Corporation, San Diego, California 92121
and
Robert J. Adams†
Raytheon Missile Systems, Tucson, Arizona 85706

Introduction

THE concepts of gain and phase margin for single-input single-output (SISO) systems are well defined and understood. In Ref. 1 Bar-on and Jonckheere extended these SISO concepts to multivariable systems. In this Note, their definition for multivariable phase is used to develop a block diagonal phase margin for systems that are subjected to diagonal perturbations which contain two pure phase uncertainty matrices. The new margin guarantees that a stable, closed-loop, multi-input multi-output system is robust to all phase uncertainties in the feedback path that have the specified structure with phase less than the block diagonal multivariable phase margin. Such uncertainties can arise in real-world systems when multiple sensors that measure different quantities are used in feedback loops. An example for a tail-controlled guided missile is presented to illustrate the new definitions and the subsequent analysis.

Background

For multivariable systems pure phase uncertainties are modeled as unitary matrices. In Bar-on and Jonckheere[1] the phase of a unitary matrix $U = e^{\Sigma}$, where Σ is a skew Hermitian matrix, is defined to be

$$\text{phase}(U) = \max|\arg[\lambda(U)]| = \max|\lambda(\Sigma)|$$

where $\lambda(\cdot)$ refers to the eigenvalues of the matrix $(\cdot)$. For nominally stable closed-loop systems, $L(s)$, destabilizing pure phase uncertainties can exist at frequencies only in the gain crossover region Ω, which is defined as

$$\Omega = \{\omega : \sigma_1[L(j\omega)] \geq 1, \sigma_i[L(j\omega)] \leq 1, \text{for some } i\} \tag{1}$$

Here $\sigma_i(\cdot)$ is the ith singular value of the matrix $(\cdot)$, and $i = 1$ corresponds to the maximum singular value. When $L(s)$ is a SISO transfer function, Eq. (1) indicates that the frequencies in the gain crossover region occur when $|L(j\omega)| = 1$. This shows that Ω is the multivariable analog of the SISO gain crossover frequency. At frequencies outside of the gain crossover region, the multivariable phase margin is considered to be infinite as there does not exist a pure phase uncertainty matrix that will destabilize the system.

The set of all destabilizing unitary perturbations is defined to be

$$D = \{\Delta : \det[I + L(j\omega)\Delta] = 0, \Delta \in \boldsymbol{U}(n, \mathbb{C}), \omega \in \Omega\}$$

Thus, any $\Delta \in D$ will push the multivariable Nyquist diagram of $L(j\omega)\Delta$ through the origin so that the closed-loop system will have poles on the $j\omega$ axis. The multivariable phase margin is defined to be the phase of the smallest phase destabilizing unitary perturbation in the gain crossover region. This can be expressed as

$$PM[L(s)] = \min_{\Delta \in D}\{\max(|\arg[\lambda(\Delta)]|)\}$$

Closed-loop stability is guaranteed for any unitary perturbation in the feedback path whose phase is less than $PM[L(s)]$. Furthermore, there exists a pure phase perturbation with phase equal to the phase margin of $L(s)$ that will destabilize the closed-loop system.

Numerical routines to calculate the gain crossover region and to determine the multivariable phase margin are described in Ref. 1. These routines are based upon solving a nonlinear constrained optimization problem. In Ref. 2 Bar-on and Grasse prove that these methods converge to the global solution for the multivariable phase margin.

Two-Block Multivariable Phase Margin

It is well known that the structured singular value can be used to measure robustness of systems that contain multiple sources of uncertainty (for example, see Refs. 3 and 4). Typically, the uncertainty sources are of varying size and can be static and/or dynamic with a norm bound. Robustness of the system is determined by reflecting all uncertainty sources to a single reference location to form a block diagonal uncertainty matrix and then determining the structured singular value of the corresponding, newly formed, interconnected transfer function matrix.[5] The end result of this type of analysis is a measure of the smallest size of structured uncertainty that will destabilize the system; the phase of the uncertainty is not considered. In this Note we consider the impact of structured, pure phase uncertainty on a nominally closed-loop stable multivariable system.

Definition 1: The structured multivariable phase margin is the phase of the smallest phase block diagonal uncetainty matrix that destabilizes the nominally closed-loop stable transfer function matrix.

In this Note the structured multivariable phase margin is developed for systems $L(s)$ that are nominally closed-loop stable and are subjected to two pure phase uncertainty sources in the feedback path. As seen in the following section, such pure phase uncertainty could easily arise when isolated sensors are used to measure different quantities. The specific structure that is considered is given by

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \tag{2}$$

where $\Delta_1 \in \boldsymbol{U}(n, \mathbb{C})$, $\Delta_2 \in \boldsymbol{U}(m, \mathbb{C})$. Because the uncertainty matrix defined in Eq. (2) is block diagonal, its phase is given by

$$\text{phase}(\Delta) = \max\{\text{phase}(\Delta_1), \text{phase}(\Delta_2)\} \tag{3}$$

Received 18 March 2002; revision received 11 July 2002; accepted for publication 27 August 2002.

*Senior Scientist, Technology Research Group, MS-C4, 10260 Campus Point Drive; jonathan.r.bar-on@saic.com.

†Principal Systems Engineer, Autopilot Department, Guidance Navigation and Control Center, 1151 E. Hermans Rd., Bldg. 805 MS M4; rjadams@raytheon.com.

The remainder of this section shows that the structured multivariable phase margin can be determined by solving a constrained optimization problem over a collection of frequencies. That is, if Ω_2 is the collection of all frequencies where there exists a destabilizing, structured, pure phase uncertainty matrix then the structured multivariable phase margin is the minimum solution of the following constrained optimization problem:

$$P1: \begin{cases} \min(\max\{\text{phase}(\Delta_1),\ \text{phase}(\Delta_2)\}) \\ \text{s.t.}\quad v = -L(j\omega)z \\ \quad \|v\|_2 = 1 \\ \quad \|z\|_2 = 1 \\ \quad \omega \in \Omega_2 \end{cases} \tag{4}$$

Calculation of the structured multivariable phase margin requires that one determine all frequencies in the set Ω_2. To do this, recognize that the smallest phase structured Δ that will destabilize $L(s)$ just pushes the multivariable Nyquist diagram through the origin. That is,

$$\det[I + L(j\omega)\Delta] = 0 \tag{5}$$

for some $\omega = \hat{\omega} \in \mathbb{R}^+$. Equation (5) implies that at $\hat{\omega}$ there exists a unit vector v such that $v = -L\Delta v$. Define $z = \Delta v$; clearly z is also a unit vector and

$$v = -Lz \tag{6}$$

Let

$$L(j\hat{\omega}) = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \qquad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \qquad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where $L_{11} \in \mathbb{C}^{n\times n}$, $L_{12} \in \mathbb{C}^{n\times m}$, $L_{21} \in \mathbb{C}^{m\times n}$, and $L_{22} \in \mathbb{C}^{m\times m}$. Substituting into Eq. (6) yields

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} L_{11}z_1 + L_{12}z_2 \\ L_{21}z_1 + L_{22}z_2 \end{bmatrix} \tag{7}$$

By definition of z, and because Δ is block diagonal,

$$z_1 = \Delta_1 v_1, \qquad z_2 = \Delta_2 v_2$$

Furthermore, as Δ_1 and Δ_2 are both unitary matrices

$$\|v_1\|_2 = \|z_1\|_2 \tag{8}$$

$$\|v_2\|_2 = \|z_2\|_2 \tag{9}$$

Substituting from Eq. (7) for v_1 into Eq. (8) yields

$$z_1^* z_1 = (L_{11}z_1 + L_{12}z_2)^*(L_{11}z_1 + L_{12}z_2)$$

$$z_1^* z_1 = \left(z_1^* L_{11}^* + z_2^* L_{12}^*\right)(L_{11}z_1 + L_{12}z_2)$$

$$0 = z_1^*\left(L_{11}^* L_{11} - I\right)z_1 + z_2^* L_{12}^* L_{12} z_2 + z_1^* L_{11}^* L_{12} z_2 + z_2^* L_{12}^* L_{11} z_1 \tag{10}$$

Similarly, substituting from Eq. (7) for v_2 into Eq. (9) yields

$$0 = z_1^* L_{21}^* L_{21} z_1 + z_2^*\left(L_{22}^* L_{22} - I\right)z_2 + z_1^* L_{21}^* L_{22} z_2 + z_2^* L_{22}^* L_{21} z_1 \tag{11}$$

Defining the matrices

$$A = \begin{bmatrix} L_{11}^* L_{11} - I & L_{11}^* L_{12} \\ L_{12}^* L_{11} & L_{12}^* L_{12} \end{bmatrix}$$

$$B = \begin{bmatrix} L_{21}^* L_{21} & L_{21}^* L_{22} \\ L_{22}^* L_{21} & L_{22}^* L_{22} - I \end{bmatrix} \tag{12}$$

enables Eqs. (10) and (11) to be expressed in matrix form as

$$0 = z^* A z \tag{13}$$

$$0 = z^* B z \tag{14}$$

Any vector z satisfying Eqs. (13) and (14) is said to be a common isotropic vector for the matrices A and B (Ref. 6). What has now been shown is that if Δ pushes the multivariable Nyquist diagram through the origin at the frequency $\hat{\omega}$, then the matrices A and B have a common isotropic vector.

It is now necessary to show that if the matrices A and B have a common isotropic vector, then at the frequency $\hat{\omega}$ there exists a block diagonal Δ that pushes the multivariable Nyquist diagram through the origin. Suppose that matrices A and B have a common isotropic vector z, and, without loss of generality, assume that z is a unit vector. Let the vector v be defined by $v = -Lz$ so that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} L_{11}z_1 + L_{12}z_2 \\ L_{21}z_1 + L_{22}z_2 \end{bmatrix}$$

Now,

$$v_1^* v_1 = (L_{11}z_1 + L_{12}z_2)^*(L_{11}z_1 + L_{12}z_2)$$

$$v_1^* v_1 = z_1^* L_{11}^* L_{11} z_1 + z_2^* L_{12}^* L_{12} z_2 + z_1^* L_{11}^* L_{12} z_2 + z_2^* L_{12}^* L_{11} z_1 \tag{15}$$

and

$$v_2^* v_2 = (L_{21}z_1 + L_{22}z_2)^*(L_{21}z_1 + L_{22}z_2)$$

$$v_2^* v_2 = z_1^* L_{21}^* L_{21} z_1 + z_2^* L_{22}^* L_{22} z_2 + z_1^* L_{21}^* L_{22} z_2 + z_2^* L_{22}^* L_{21} z_1 \tag{16}$$

As z is an isotropic vector of both A and B, it satisfies Eqs. (13) and (14) or, equivalently, Eqs. (10) and (11). Substitution into Eqs. (15) and (16) yields

$$v_1^* v_1 = z_1^* z_1 \tag{17}$$

$$v_2^* v_2 = z_2^* z_2 \tag{18}$$

Proposition 1: Given any two vectors x and $y \in \mathbb{C}^n$ with equal magnitude, there always exists a unitary matrix $U \in U(n, \mathbb{C})$ such that $y = Ux$.

Proof: Define the unit vectors $\hat{x} = x/|x|$ and $\hat{y} = y/|y|$. Using Gram–Schmidt orthogonalization an orthonormal basis for $\mathbb{C}^n$ can be formed where the first vector in the basis is $\hat{x}$. Let the columns of the matrix $T \in U(n, \mathbb{C})$ be the corresponding vectors of this basis:

$$T = [\hat{x} \quad t_1 \quad \cdots \quad t_{n-1}] \Rightarrow TT^* = I$$

so that

$$\hat{x} = Te_1 \tag{19}$$

Similarly, another orthonormal basis for $\mathbb{C}^n$ can be formed where the first vector in the basis is $\hat{y}$. Let the columns of the matrix $P \in U(n, \mathbb{C})$ be the corresponding vectors of this basis:

$$P = [\hat{y} \quad p_1 \quad \cdots \quad p_{n-1}] \Rightarrow PP^* = I$$

so that

$$\hat{y} = Pe_1 \tag{20}$$

Combining Eqs. (19) and (20) yields

$$\hat{y} = PT^* \hat{x}$$

Because $U(n, \mathbb{C})$ is an algebraic group, the matrix $U = PT^* \in U(n, \mathbb{C})$. Substituting for $\hat{x}$ and $\hat{y}$ and recalling that $|x| = |y|$ yields

$$y = Ux$$

which completes the proof. □

When applied to Eqs. (17) and (18), the preceding proposition states that one can always find unitary matrices U_1 and U_2 such that

$$z_1 = U_1 v_1, \qquad z_2 = U_2 v_2$$

Defining

$$\Delta = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$$

where $\Delta \in \boldsymbol{U}(n+m, \mathbb{C})$ results in $z = \Delta \boldsymbol{v}$. Thus,

$$\boldsymbol{v} = -Lz, \qquad \boldsymbol{v} = -L\Delta\boldsymbol{v}$$

$$(I + L\Delta)\boldsymbol{v} = 0, \qquad \det(I + L\Delta) = 0$$

This shows that if the matrices A and B have a common isotropic vector, then it is always possible to find a unitary block diagonal matrix Δ which will push the multivariable Nyquist diagram of $L(j\omega)\Delta$ through the origin. In fact, it has now been shown that a necessary and sufficient condition for the existence of a diagonal two-block, pure phase uncertainty matrix which will push the multivariable Nyquist diagram through the origin at a frequency $\hat{\omega}$ is the existence of a common isotropic vector for the matrices A and B. Define the set Ω_2 to be the collection of all frequencies where the matrices A and B have a common isotropic vector

$$\Omega_2 = \{\omega : \exists \text{ a vector } z \text{ such that } z^* Az = z^* Bz = 0\}$$

This set is the generalization of the gain crossover region for the block diagonal case containing two pure phase uncertainty matrices. The following proposition describes how the set Ω_2 can be calculated.

Proposition 2: Consider the Hermitian matrices A and $B \in \mathbb{C}^{pxp}$. There exists a common isotropic vector for the matrices A and B if and only if 0 is contained in the numerical range of the matrix $C = A + iB$.

Proof: The existence of a common isotropic vector is equivalent to the existence of a nonzero vector $z \in \mathbb{C}^p$ such that $z^*Az = z^*Bz = 0$. Without loss of generality, it is assumed that z is a unit vector. (Sufficiency) Suppose 0 is contained in the numerical range of the matrix $C = A + iB$. Then $\exists$ a unit vector z such that $z^*Cz = 0$. Substituting for C yields

$$z^*(A + iB)z = 0, \qquad z^*Az + iz^*Bz = 0 \tag{21}$$

As A and B are both Hermitian, z^*Az and z^*Bz are real for any vector z. Thus, to satisfy Eq. (21) $z^*Az = 0$ and $z^*Bz = 0$.

(Necessity) Suppose now that z is a unit vector such that $z^*Az = z^*Bz = 0$. Then $z^*Az + \alpha z^*Bz = 0$ for any $\alpha \in \mathbb{C}$. Thus, $z^*(A + \alpha B)z = 0$ and taking $\alpha = i$ completes the proof. □

To determine the structured multivariable phase margin for phase uncertainties described by Eq. (2), the phase of the smallest phase two-block perturbation that destabilizes $L(s)$ needs to be calculated at each frequency $\omega \in \Omega_2$. The block diagonal phase margin will then be the minimum phase of these smallest phase two-block perturbations taken over the frequencies in Ω_2. The constrained optimization problem given in Eq. (4) is used to calculate the smallest phase two block perturbation at discrete frequencies in Ω_2. In this problem the cost function is the phase of the block diagonal uncertainty at each $\omega \in \Omega_2$, and the constraints guarantee that the multivariable Nyquist diagram will pass through the origin [recall Eqs. (5) and (6) and the comments in between]. Substituting for $\boldsymbol{v}$ allows one to write the norm constraints as $z^*L^*Lz = 1$ and $z^*z = 1$. The block diagonal structure of the perturbation results in two additional constraints that are given by Eqs. (8) and (9). As just shown, these constraints can be expressed using Eqs. (13) and (14) where the matrices A and B are defined in Eq. (12). Note that $A + B + I = L^*L$ so that the first constraint $z^*L^*Lz = 1$ is redundant when the last three constraints are satisfied. Thus, to calculate the block diagonal phase margin the following constrained optimization problem needs to be solved at each ω in Ω_2:

$$P2: \begin{cases} \min(\max\{\text{phase}(\Delta_1), \text{phase}(\Delta_2)\}) \\ \text{s.t.} \quad z^*z = 1 \\ \qquad z^*Az = 0 \\ \qquad z^*Bz = 0 \end{cases} \tag{22}$$

Recall that $z = \Delta\boldsymbol{v}$ so that as described in Ref. 1 the phase of each Δ_i block is given by

$$\text{phase}(\Delta_i) = \cos^{-1}\left(\frac{\boldsymbol{v}_i^* z_i + z_i^* \boldsymbol{v}_i}{2\|\boldsymbol{v}_i\|_2 \|z_i\|_2}\right)$$

Expressions for the phase of each Δ_i block in terms of only z and L can be derived using Eqs. (6), (8), and (9). Substitution yields

$$\text{phase}(\Delta_1) = \cos^{-1}\left[\frac{-z^*Cz}{2z^*(A + I_u)z}\right]$$

$$\text{phase}(\Delta_2) = \cos^{-1}\left[\frac{-z^*Dz}{2z^*(B + I_l)z}\right]$$

where

$$C = \begin{bmatrix} L_{11}^* + L_{11} & L_{12} \\ L_{12}^* & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & L_{21}^* \\ L_{21} & L_{22}^* + L_{22} \end{bmatrix}$$

$$I_u = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad I_l = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

Finally, as z satisfies the constraint at an extremum

$$2z^*(A + I_u)z = 2z^*I_uz, \qquad 2z^*(B + I_l)z = 2z^*I_lz$$

and $(P2)$ can be written as

$$P3: \begin{cases} \min\left(\max\left\{\cos^{-1}\left(\dfrac{-z^*Cz}{2z^*I_uz}\right), \cos^{-1}\left(\dfrac{-z^*Dz}{2z^*I_lz}\right)\right\}\right) \\ \text{s.t.} \quad z^*z = 1 \\ \qquad z^*Az = 0 \\ \qquad z^*Bz = 0 \end{cases} \tag{23}$$

Solutions of $(P3)$ yield the phase of the smallest phase two-block perturbation at a specific frequency in Ω_2. The minimum of these solutions taken over all frequencies in Ω_2 is the multivariable block diagonal phase margin for pure phase uncertainty matrices whose structure is defined in Eq. (2).

Example

In this section the multivariable block diagonal phase margin of a tail-controlled guided missile is examined and compared to the multivariable phase margin defined in the Background section. The system that was analyzed consisted of a six-degree-of-freedom linear missile model, four second-order actuators, three second-order rate gyroscopes, two second-order accelerometers, and a classical decoupled three-loop roll-pitch-yaw autopilot whose gains are scheduled on Mach and altitude. A block diagram of the missile system, including the four pure phase perturbations analyzed in this section, appears in Fig. 1. Additional details on how to linearize the six-degree-of-freedom equations governing missile dynamics can be found in Ref. 7. The flight condition chosen for analysis was Mach 2, 20,000 ft, aerodynamic roll angle $\phi_A = 0$ deg, and total angle of attack $\alpha_T = 16$ deg.

The multivariable block diagonal phase margin was calculated by breaking the gyro and accelerometer feedback loops. The partitioning for the pure phase uncertainty matrix was a 2×2 upper block associated with the accelerometers and a 3×3 lower block associated with the gyros. The resulting linear system consisted of 41 states with five inputs and five outputs. For comparative purposes the multivariable phase margin was also calculated for three other systems using a single, pure phase, uncertainty matrix. These additional systems were structured as follows:

1) The two accelerometer loops in the feedback path were broken while the gyro loops were closed; this corresponds to the upper 2×2 block phase uncertainty matrix.

2) The three gyro loops in the feedback path were broken while the accelerometer loops were closed; this corresponds to the lower 3×3 block phase uncertainty matrix.

3) Both the gyro and accelerometer loops were broken; this corresponds to a fully populated 5×5 pure phase uncertainty matrix inserted in the feedback path.

Each of these three systems contained 41 states, and, respectively, the number of inputs and outputs were two inputs/two outputs, three

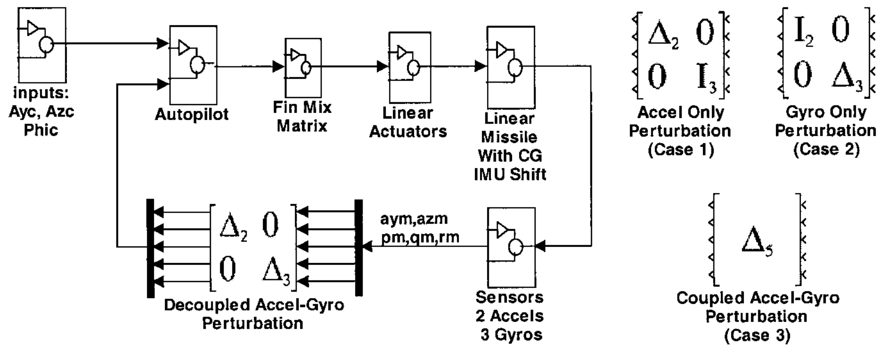

Fig. 1 Linear six-degree-of-freedom model and the four multivariable pure phase perturbations analyzed.

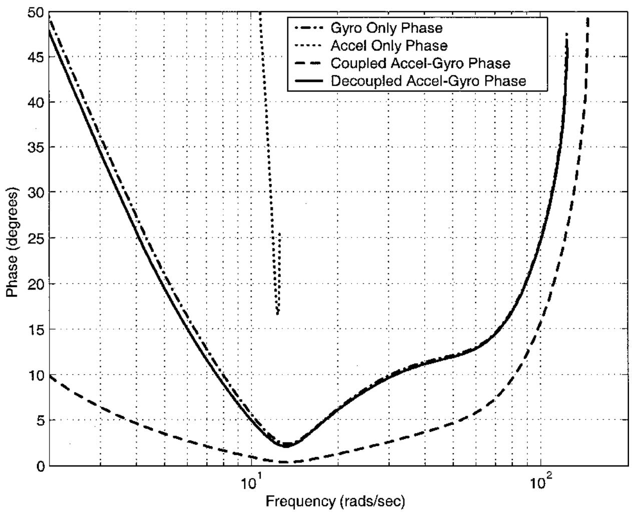

Fig. 2 Comparison of the two-block phase to the three single-block phase cases.

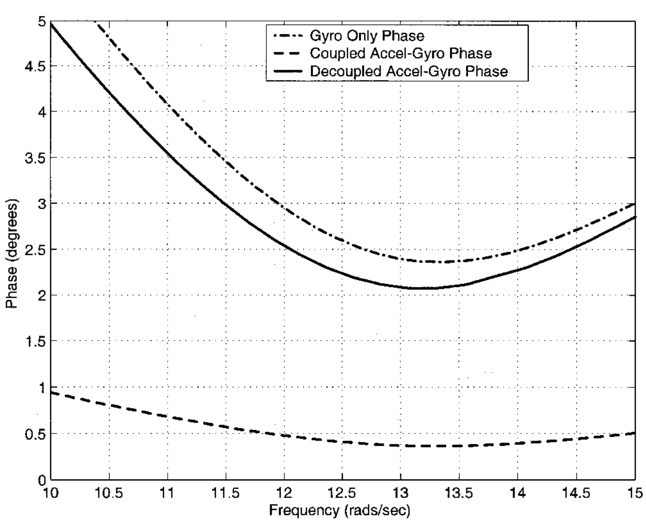

Fig. 3 Comparison of the two-block phase margin to the gyro-only and fully coupled phase margins.

inputs/three outputs, and five inputs/five outputs. Note that the physical interpretation of the pure phase perturbation as a rotation matrix coupled with computational delays for the third case is untenable. This fact was the primary motivation behind the current work.

Figure 2 contains the phase of the smallest phase destabilizing block diagonal pure phase perturbation over the frequency range Ω_2. This figure also contains the phase of the smallest destabilizing pure phase perturbations associated with the three cases just described for the frequency range $2 \leq \omega \leq 200$ rad/s. The gain crossover region for each of these systems is different because, as defined by Eq. (1), Ω is a function of the spread of the singular values of the loop transfer function matrix $L(s)$ across 1. As seen in Fig. 2, the block diagonal phase for the decoupled accelerometer gyro is bounded from below by the fully coupled accelerometer-gyro phase [case 3)] and from above by the gyro-only phase [case 2)]. The bounding from above is expected as the gyro-only case and the accelerometer-only case each contain a single pure phase uncertainty matrix with, respectively, the accelerometer or gyro loops closed with identity in the feedback path. The closeness of the block diagonal phase to the gyro-only phase is not surprising because decoupled three-loop autopilots are more sensitive to perturbations in rate feedback paths than to perturbations in acceleration feedback paths. The reason for this sensitivity is that the autopilot uses rate feedback to add stability to the closed-loop system, whereas acceleration feedback is used to meet tracking requirements. The block diagonal phase is bounded from below by the fully populated phase perturbation as this perturbation covers a larger uncertainty space and contains coupling that is not physically tractable. Figure 3 shows a comparison of the phase of the smallest destabilizing block diagonal pure phase perturbation to the smallest destabilizing pure phase perturbations associated with cases 2 and 3 for $10 \leq \omega \leq 15$ rad/s. This plot indicates that the block diagonal phase margin for this case is 2.08 deg and occurs at 13.31 rad/s while the multivariable phase margin for the gyro-only case is 2.36 deg and occurs at 13.36 rad/s.

Conclusions

In this Note, the multivariable phase margin for a block diagonal, pure phase uncertainty matrix is developed. The structure of the perturbation is restricted to two pure phase uncertainty matrices. As shown in the example of a fully coupled six-degree-of-freedom linear representation of a guided missile, such perturbations can arise when multiple sensors measuring different quantities are used in feedback loops. The example illustrates that the phase of the smallest destabilizing block diagonal phase uncertainty matrices is bounded from below by the phase of the fully populated destabilizing pure phase uncertainty matrices and from above by the phase of the upper and lower pure phase uncertainty matrices. For the missile example the bounding from above with respect to the gyro-only phase uncertainty is tight because the autopilot uses rate feedback for stability augmentation.

References

[1]Bar-on, J. R., and Jonckheere, E. A., "Phase Margins for Multivariable Control Systems," *International Journal of Control*, Vol. 52, No. 2, 1990, pp. 485–498.

[2]Grasse, K., and Bar-on, J. R., "Regularity Properties of the Phase for Multivariable Systems," *SIAM Journal of Control and Optimization*, Vol. 35, No. 4, 1997, pp. 1366–1386.

[3]Balas, G. J., Doyle, J. C., Glover, K., Packard, A., and Smith, R., *μ-Analysis and Synthesis Toolbox*, The Mathworks, Inc., Natick, MA, 1991, pp. 4.1–4.86.

[4]Skogestad, S., and Postlethwaite, I., *Multivariable Feedback Control Analysis and Design*, Wiley, New York, 1996, pp. 309–322.

[5]Zhou, K., Doyle, J. C., and Glover, K., *Robust and Optimal Control*, Prentice–Hall, Upper Saddle River, NJ, 1996, pp. 271–286.

[6]Horn, R. A., and Johnson, C. R., *Topics in Matrix Analysis*, Cambridge Univ. Press, Cambridge, England, U.K., 1991, pp. 69–73.

[7]Bar-on, J. R., and Adams, R. J., "Linearization of a Six Degree of Freedom Missile for Autopilot Analysis," *Journal of Guidance, Navigation, and Control*, Vol. 21, No. 1, 1998, pp. 184–187.